FULLY MULTI-QUBIT ENTANGLED STATES

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Abstract

We investigate the properties of different levels of entanglement in graph states which correspond to connected graphs. Combining the operational definition of graph states and the postulates of entanglement measures, we prove that in connected graph states of $N > 3$ qubits there is no genuine three-qubit entanglement. For certain classes of graph states, all genuine k-qubit entanglement, $2 \leq k \leq N - 1$, among every k qubits vanishes. These results about connected graph states naturally lead to the definition of fully multi-qubit entangled states. We also find that the connected graph states of four qubits is one but not the only one class of fully four-qubit entangled states.

Keywords: Graph States, Multi-qubit Entanglement, Quantum Computation

1. Introduction

The trend of quantum information processing is to implement large scale quantum computation with many qubits [1]. One prospective proposal is the one-way quantum computation model, based on some special kind of multi-particle entangled states and single qubit measurements [2]. The universal resource in one-way quantum computer is the so called graph states that correspond to mathematical graphs [3], where the vertices of the graph play the role of quantum spin systems and edges represent Ising interactions. Graph states also have applications in quantum communication of many users, e.g. open destination quantum teleportation [4]. Moreover, various quantum error correcting codes for protecting quantum information against decoherence are also graph states [5].

On the other hand, it is well known that entanglement is the most fascinating feature of quantum mechanics. Very recently, entanglement in interacting many-body systems becomes an increasing important concept in condensed matter physics, such as quantum phase transitions [6], superconductivity and fractional quantum Hall effect [7]. However, the structure and nature of entanglement in multi-particle entangled states is not very

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clear now. The most obstacle is that there is no known measure which can completely characterize the entanglement of multi-particle entangled states. Therefore, the study of entanglement properties of the special significant multi-particle entangled states - graph states is a very important and interesting topic [8, 9, 10, 11, 12].

In Refs.[9], Hein *et al* characterize and quantify the genuine multi-particle entanglement of graph states by the Schmidt measure. They provide the upper and lower bounds of the Schmidt measure [13] in graph theoretical terms. In this paper, we investigate the entanglement properties of graph states from the viewpoint of different levels of entanglement. The main result is that, using the operational description of graph states and the fact that entanglement measures always decrease under local operations and classical communications (LOCC), we present a simple proof that in general connected graph states of $N > 3$ qubits, there is no genuine three-qubit entanglement. Moreover, for some classes of connected graphs states, all genuine k-qubit entanglement, $2 \leq k \leq N - 1$, among every k qubits always vanishes. These results explicitly demonstrate that graph states are indeed a kind of fully multi-qubit entangled states. In addition, we find that the connected graph states of four qubits is only one class of fully multi-qubit entangled states. We construct different kinds of fully multi-qubit entangled states that are not local unitary equivalent to connected graph states of four qubits.

2. Graph states

Each mathematical (undirected, finite) *graph* is denoted as [14]

$$G = (V, E) \tag{1}$$

where the finite set $V \subset \mathcal{N}$ is the set of vertices, and the set $E \subset [V]^2$ is the set of edges. In the context of graph states, people restrict to the simple graphs, which contain neither edges connecting vertices with itself nor multiple edges. Given a subset of vertices $S \subset V$, we can define the *subgraph* generated by S as $G_S = (S, E_S)$, where $E_S \subset E$, and for every edges $\{a, b\} \in E$, if and only if $a, b \in S$ then $\{a, b\} \in E_S$.

For a given vertex $a \in V$, its *neighborhood* $N_a \subset V$ is defined as the set of vertices adjacent to the given vertex a, i.e. the set of vertices $b \in V$ for which $\{a, b\} \in E$. For two vertices $a, b \in V$, we say a and b is connected if there exists an ordered list of vertices $a = a_1, a_2, \cdots, a_{n-1}, a_n = b$ such that for all i, $(a_i, a_{i+1}) \in E$. If any two $a, b \in V$ are connected, the graph is a *connected graph*, otherwise it is a disconnected graph which can be viewed as a collection of several separate connected subgraphs.

Graph states that correspond to a mathematical graphs $G = (V, E)$ is a certain pure quantum state on the Hilbert space $H = (\mathcal{C}^{\in})^{\otimes \mathcal{N}}$, where $N = |V|$ is the number of the vertices. For every vertex $a \in V$ of the graph $G = (V, E)$, one can define a Hermitian operator,

$$K_G^a = X_a \bigotimes_{b \in N_a} Z_b \tag{2}$$

where the matrices X_a, Y_a and Z_a are Pauli matrices, the lower index specifies the qubit on which the operators acts. The *graph state* $|G\rangle$ associated with the graph $G = (V, E)$ is the unique $n-$qubit state fulfilling

$$K_G^a |G\rangle = |G\rangle \tag{3}$$

The graph state $|G\rangle$ can be obtained by applying a sequence of unitary two-qubit operations to the initial state $|+\rangle^{\otimes N}$ as follows,

$$|G\rangle = \prod_{\{a,b\}\in E} U_{ab}|+\rangle^{\otimes N} \tag{4}$$

where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, and $|0\rangle, |1\rangle$ are eigenvectors of Z with eigenvalues ± 1. The unitary two-qubit operation U_{ab} is a controlled Z on qubits a and b, i.e. $U_{ab} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$. We note that theses unitary two-qubit operations commute with each other. Therefore, we can adopt different orders of the sequence of U_{ab} to the initial state $|+\rangle^{\otimes N}$ and yield the same graph state $|G\rangle$. This property is the key point in our following proof.

As discussed above, a disconnected graph can be viewed as a collection of several separate connected subgraphs. Therefore, a disconnected graph state is just a product state of the corresponding connected subgraph states. Without loss of generality, it is sufficient for us to consider only the connected graph states here. The entanglement structure in multi-qubit entangled states is much more complex than the situation of two-qubit entangled states. For pure states of N qubits, there are different levels of genuine k-qubit entanglement, $2 \leq k \leq N$, which is shared among all the k qubits.

In Refs. [10], Hein *et al* show that there is no 2-qubit entanglement between any two qubits in general N-vertex connected graph states with $N \geq 3$ by examining the properties of reduced density matrices. However, there is no exact results about general genuine k-qubit entanglement for $3 \leq k \leq N-1$. One reason is that unlike 2-qubit entanglement entanglement [15], there are few well defined genuine multi-qubit entanglement measures [16, 17, 18, 19, 20, 21, 22, 23], especially for general multi-qubit mixed states. For a natural entanglement measure, it should satisfy several necessary conditions, such as invariant under local unitary operations, vanish for separable states, and decrease on average under LOCC. In the following, using the operational definition of graph states and the postulates of entanglement measures, we first prove that genuine three-qubit entanglement vanish in connected graph states of four qubits. Then we will generalize our results to arbitrary genuine k-qubit entanglement.

3. Genuine three-qubit entanglement in graph states

In this section, we will investigate genuine three-qubit entanglement among every three qubits in connected graph states. Genuine three-qubit entanglement is the special kind of entanglement which critically involves all of the three qubits. For example, in GHZ state of three qubits $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, the state becomes separable if any one qubit is lost, which means that the entanglement in GHZ state is the only global property shared by all of the three qubits. However, in W state [24] of three qubits $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, two-qubit entanglement will still exist when one qubit is neglected. One well defined measure for genuine three-qubit entanglement is the square root of CKW tangle [25] proposed by Wootters *etc*, of which $\tau(|GHZ\rangle) = 1$ and $\tau(|W\rangle) = 0$.

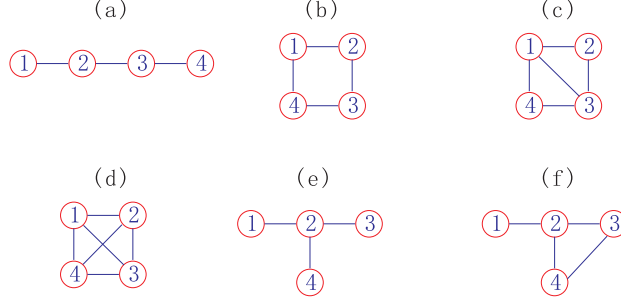*Lemma 1: Genuine three-qubit entanglement vanishes in connected graph states of four qubits.*

Fig. 1. Six classes of four vertices connected graphs that are nonequivalent under graph isomorphisms.

Proof: There are six classes of four vertices connected graphs that are nonequivalent under graph isomorphisms as depicted in Fig. (1). For each graph state $|G\rangle_{1234}$, we can write the reduced density matrix ρ_{ijk} of every three qubit $i, j, k = 1, 2, 3, 4$. By exploiting some skills, it is easy for us to construct a special pure states decomposition of ρ_{ijk} as $\rho_{ijk} = \sum_i |\phi_i'\rangle\langle\phi_i'|$, where $|\phi_i'\rangle$ are separable. For example, in the graph state $|G_d\rangle_{1234}$ corresponding to Fig 1.(d), $\rho_{123} = \frac{1}{2}(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)$, where $|\phi_1\rangle = \frac{1}{2}(|+00\rangle + |-01\rangle + |-10\rangle - |+11\rangle)$ and $|\phi_2\rangle = \frac{1}{2}(|-00\rangle - |+01\rangle - |+10\rangle - |-11\rangle)$. The special pure states decomposition for ρ_{123} is $\rho_{123} = \sum_{i=1}^{2} |\phi_i'\rangle\langle\phi_i'|$, where $(|\phi_1'\rangle, |\phi_2'\rangle)^T = U(|\phi_1\rangle, |\phi_2\rangle)$, and $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. It is obvious that $|\phi_1'\rangle$ and $|\phi_2'\rangle$ are separable for bipartition $1|23$. Actually, we could show that the square root of CKW tangle $\tau(\rho_{123}) = 0$ in a similar way. Therefore we conclude that genuine three-qubit entanglement vanishes for every three qubits.□

It should be emphasized that we only need to consider 2 equivalence classes under local Clifford (LC) operations [10]. One class includes Fig 1.(a), (b), (c), and (f), the other class includes Fig 1. (d) and (e) [26]. Based on lemma 1, we present the following theorem about genuine three-qubit entanglement in general connected graph states of more than three qubits.

Theorem 1: There is no genuine three-qubit entanglement in general connected graph states of more than three qubits.

Proof: We examine genuine three-qubit entanglement among every three qubits. Without loss of generality, we can denote these three qubits as 1, 2, and 3. According to whether the subgraph $G_{\{1,2,3\}} = (\{1, 2, 3\}, E_{\{1,2,3\}})$ is connected, there are two kinds of situations.

(a). The subgraph $G_{\{123\}}$ is connected. Since the number of qubits $N > 3$ and the corresponding graph is connected, there must exist one qubit 4 which make the subgraph $G_{\{1,2,3,4\}} = (\{1, 2, 3, 4\}, E_{\{1,2,3,4\}})$ is also connected, i.e. it is one class of graph depicted in Fig 1. (a1). In the first step to obtain the graph state $|G\rangle$, we get the

state $|\psi\rangle = \prod_{\{i,j\}\in E_{\{1,2,3,4\}}} U_{ij}|+\rangle^{\otimes N}$, i.e. $|\psi\rangle = |G_{\{1,2,3,4\}}\rangle|+\rangle^{\otimes N-4}$, where $|G_{\{1,2,3,4\}}\rangle$
is the connected graph states of four qubits, which leads to that genuine three-qubit
entanglement in ρ_{123} is 0 for $|\psi\rangle$ according to lemma 1. (a2). In the second step, we
apply unitary two-qubit operations related to qubit 1, 2 and 3 to $\psi\rangle$ and obtain $|\psi'\rangle = \prod_{i\in V-\{1,2,3,4\},j\in\{1,2,3\},\{i,j\}\in E} U_{ij}|\psi\rangle$. The effects of each block of operations $\prod_{j\in\{1,2,3\},\{i,j\}\in E} U_{ij}$
for $i \in V - \{1,2,3,4\}$ on ρ_{123} can be characterize by the superoperator $\varepsilon(\rho_{123}) = \frac{1}{2}\rho_{123} + \frac{1}{2}(u_1 \otimes u_2 \otimes u_3)\rho_{123}(u_1 \otimes u_2 \otimes u_3)^\dagger$, where u_1, u_2, $u_3 = I_2$ or Z. It de-
scribes a certain local operation on qubit 1, 2 and 3. Since any entanglement measure
should be an entanglement monotone function [15], thus genuine three-qubit entangle-
ment in ρ_{123} is 0 for $|\psi'\rangle$. (a3). In the last step, we apply the remain unitary two-qubit
operations independent on qubit 1, 2 and 3 to $|\psi'\rangle$ and obtain the final graph state
$|G\rangle = \prod_{i,j\in V-\{1,2,3\},\{i,j\}\in E} U_{ij}|\psi'\rangle$. In this step, ρ_{123} is unchanged. Therefore, we con-
clude that genuine three-qubit entanglement in ρ_{123} vanishes for $|G\rangle$.

(b) The subgraph $G_{\{1,2,3\}}$ is disconnected. In this situation, we first get $|\psi\rangle = \prod_{\{i,j\}\in E_{\{1,2,3\}}}$
$U_{ij}|+\rangle^{\otimes N}$. Since $G_{\{1,2,3\}}$ is disconnected, it is obvious that the state $|\psi\rangle$ is separable.
The following steps are similar to the above situation (a). Therefore, we also obtain that
genuine three-qubit entanglement in ρ_{123} vanishes for the state $|G\rangle$, and theorem 1 is
proved. □

Our results imply that for general connected graph states of more than three qubits,
if we consider the reduced states of every three qubits by tracing out the other qubits, no
genuine three-qubit entanglement exists. In other words, entanglement in these graph
states is the properties that critically involves more than three qubits which is similar to
the case of GHZ states.

4. General genuine k-qubit entanglement in graph states

To our knowledge, the problem about general multipartite entanglement is extremely
hard and remain far from clear, in particular there is no well-defined general genuine
multi-qubit entanglement measures. Thus, it is very difficult to investigate the properties
of general genuine k-qubit entanglement in graph states. Even though, for some classes
of graph states, it is still possible for us to tackle this problem.

*Theorem 2: Consider a connected graph states of N qubits, and let S be a group of m
qubits, $2 \le m \le N - 1$, the reduced density matrix ρ_S is separable for certain bipartition
if the subgraph $G[S]$ can be made disconnect by using all possible local complementations
of the graph.*

Proof: It is known that the equivalence of graphs under local complementation implies
the local Clifford equivalence of the corresponding graph states with the same entangle-
ment properties [27]. Thus, if the subgraph $G[S]$ can be made disconnect by using local
complementations of graph, the reduced density matrix ρ_S is local unitary equivalent to
an ensemble of the graph state $|G[S]\rangle$ with possible local Pauli Z operations. Therefore,
ρ_S is bipartite separable.

From the above theorem 2, it is easy for us to make the statement about the properties

of all genuine k-qubit entanglement for some classes of connected graph states.

Corollary 1: All genuine k-qubit entanglement, $2 \leq k \leq N-1$, vanishes in connected graph states of N qubits which are locally equivalent to 1D cluster states, complete graph and tree graph states.

Proof: According to theorem 2, for these graph states which are locally equivalent to 1D cluster states, complete graph and tree graph states, given any subset S of $2 \leq k \leq N-1$ qubits, the reduced density matrix ρ_S is separable for some bipartition, therefore any genuine k-qubit entanglement measure should be zero, which is one of the necessary conditions for entanglement measures. Thus we finish the proof of corollary 1.

The definition of genuine multipartite entanglement here is based on the idea of residual entanglement [25], which is different from others, e.g. entanglement measures based on GHZ extraction yield and in stabilizer formalism [28, 29]. Whether the above corollary 1 is valid for general graph states remains an interesting open problem.

5. Fully multi-qubit entangled states

The connected graph states can not be written as a product form for any bipartition, and it is believed that connected graph states are genuine multi-qubit entangled states, which is supported by our result in corollary 1. Moreover, it is easy to check that the reduced density matrix of each qubit is $I/2$. With these intuitions, we could naturally define the fully multi-qubit entangled states as

A pure state of N qubits $|\psi\rangle$ is fully N-qubit entangled if it satisfies: (1) There does not exist a bipartition such that $|\psi\rangle$ is product; (2) The reduced state of each qubit is maximally mixed, i.e. $\rho_i = Tr_{1,2,\cdots,i-1,i+1,\cdots N} |\psi\rangle \langle\psi| = I/2$; (3) There is no genuine k-qubit entanglement for $2 \leq k \leq N-1$.

If $N = 2$, the above definition will reduce to maximally two-qubit entangled states. We note that stronger and slight different definitions of maximal multipartite entanglement appeared in Refs [30, 31]. However, the criterions we propose above are from the viewpoint of different levels of entanglement in multi-qubit entangled states, and is strongly motivated by the important class of multi-qubit entangled states, i.e. graph states. The above condition 1 means that $|\psi\rangle$ is not separable. The condition 2 stems from the complementary relations in multi-qubit entangled states [32, 33], the fact that the reduced state of each qubit is maximally mixed implies that local information is minimum, i.e. maximum entanglement. The last condition is introduced according to different levels of entanglement structure in multi-qubit entangled states. For example, in general N-qubit GHZ states there is only genuine N-qubit entanglement, which is shared among all of the N qubits. However, in general N-qubit W states, there are only two-qubit entanglement, i.e. shared only between pairs of qubits. In this sense, N-qubit GHZ states are fully N-qubit entangled states, while N-qubit W states are not.

The connected graph states of four qubits $|G_4\rangle$ is a kind of fully four-qubit entangled states. However, it not the only class of fully four-qubit entangled states. A generic pure state of four qubits can always be transformed to the normal form state by the determinant 1 SLOCC (stochastic local operations and classical communication) operations [34, 35], $G_{abcd} = \frac{a+d}{2}(|0000\rangle + |1111\rangle) + \frac{a-d}{2}(|0011\rangle + |1100\rangle) + \frac{b+c}{2}(|0101\rangle + |1010\rangle) + \frac{b-c}{2}(|0110\rangle + |1001\rangle)$, where a, b, c, d are complex parameters with nonnegative real part.

Without loss of generality, we could assume $A = (a + d)/2$ is a positive real number, i.e. $A = x_1 = |A|$. We denote $B = (b + c)/2 = x_2 \exp(i\phi_2), C = (a - d)/2 = x_3 \exp(i\phi_3), D = (b - c)/2 = x_4 \exp(i\phi_4)$, with $x_2 = |B|$, $x_3 = |C|$, $x_4 = |D|$. We consider those G_{abcd} that is a non-product state. The reduced density matrix of each qubit in G_{abcd} is $I/2$. The square root of CKW tangle of the mixed states obtained by tracing out one qubit of G_{abcd} is always equal to zero [34]. To ensure that pairwise entanglement also vanish in G_{abcd}, the parameters should fulfill the following conditions

$$
\begin{aligned}
2x_1 x_2 |\cos\phi_2| &\leq x_3^2 + x_4^2 \\
2x_3 x_4 |\cos(\phi_3 - \phi_4)| &\leq x_1^2 + x_2^2 \\
2x_1 x_3 |\cos\phi_3| &\leq x_2^2 + x_4^2 \\
2x_2 x_4 |\cos(\phi_2 - \phi_4)| &\leq x_1^2 + x_3^2 \\
2x_1 x_4 |\cos\phi_4| &\leq x_2^2 + x_3^2 \\
2x_2 x_3 |\cos(\phi_2 - \phi_3)| &\leq x_1^2 + x_4^2
\end{aligned}
\tag{5}
$$

Therefore, we can easily construct an explicit example of fully four-qubit entangled states

$$
|MG_4\rangle = c(|0000\rangle + |1111\rangle) + i\sqrt{\frac{1}{2} - c^2}(|0011\rangle + |1100\rangle)
\tag{6}
$$

where $0 \leq c \leq \sqrt{1/2}$. It is obvious that $|MG_4\rangle$ is not always local unitary equivalent to any four-qubit connected graph state $|G_4\rangle$. This can be verified by noting that $Tr\rho_{23}^2$ is not always the same for $|MG_4\rangle$ and $|G_4\rangle$, where ρ_{23} is the reduced density matrix of qubit 2 and 3.

6. Conclusions and Discussions

In conclusion, we have investigate the entanglement properties of graph states by calculating different levels of genuine k-qubit entanglement. In this paper, via the operational definition of graph states, and using the postulates of entanglement measures, we construct explicit proofs that there are no genuine three-qubit entanglement in general connected graph states of $N > 3$ qubits. For some class of graph states, all k-qubit entanglement vanishes for $2 \leq k \leq N - 1$. These results, together with intuitions about graph states, lead to a definition of fully multi-qubit entangled states. In addition, we find that the connected graph states of four qubits is only one class of fully multi-qubit entangled states. We present a kind of fully multi-qubit entangled states that are not always local unitary equivalent to connected graph states of four qubits. Our results demonstrate exactly that graph states are genuine multi-qubit entangled states. It may help us to gain some insight into the complex entanglement structure of multi-qubit entangled states from a new viewpoint of different levels of entanglement.

Acknowledgements

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